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1995 J. Phys. A: Math. Gen. 28 L67

(http://iopscience.iop.org/0305-4470/28/3/002)

J. Phys. A: Math. Gen. 28 (1995) L67–L72. Printed in the UK

LETTER TO THE EDITOR

Damage spreading in the Ising model with Glauber dynamics

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Received 17 November 1994

Abstract. We present accurate simulations of damage spreading for Glauber dynamics of the Ising model near the spreading threshold T_d. Without magnetic field, we find $T_d < T_c$ both in two and three dimensions. For temperatures above T_d there exists a critical magnetic field $B_d(T)$ below which the damage spreads. In all cases the critical behaviour at threshold is consistent with directed percolation.

Damage spreading [1, 3] has become a very useful tool in studying time-dependent phenomena in spin systems and in stochastic cellular automata. Here one studies two copies of the same system with slightly different initial conditions but with *identical* (pseudo-)random number sequences. The sites where the two configurations disagree are called 'damaged', and one is interested in whether the initial damage will grow (indicating a sensitive dependence on initial conditions) or will shrink (or 'heal'). In particular, using this method with heat-bath updating and following the spatial evolution of the damage, it was recently shown [4] that one can obtain very precise estimates of dynamical critical exponents in Ising models with 'model A' dynamics [5].

In the latter example, the threshold for damage spreading coincides with the Curie point T_c. Thus at the threshold there are long-range correlations in the 'undamaged' system. Alternatively, there exist cases where the undamaged system at the damage spreading transition shows no sign of any critical behaviour or any other special features. It was conjectured in [6] that in such cases this transition should be in the universality class of directed percolation (DP). This was supported in [6] by large-scale simulations of the Domany–Kinzel automaton [7] where a damage spreading transition had been found by Martins *et al* [8].

For Ising models, the heat-bath algorithm has a very special property [9] (called 'monotonicity' in [10]) which is not shared by other popular algorithms like Glauber or Metropolis (for precise definitions of these algorithms in the context of damage spreading, see [11]). Assume we have two replicas of the same lattice, let us call them A and B, and assume that we have at time $t = 0$

$$S_x^{(A)}(0) \geqslant S_x^{(B)}(0) \tag{1}$$

for each site x. Thus each 'up' spin in B is also 'up' in A, while the opposite is not necessarily true. It is straightforward to check that (1) remains true for all subsequent times, if we use the heat-bath algorithm with identical sequences of random numbers for updating A and B.

0305-4470/95/030067+06$19.50

The importance of this monotonicity for damage spreading was pointed out by Coniglio *et al* [9]. Assume that the initial configurations of A and B are identical except for a single spin at $x = 0$ for which we assume $S_0^{(A)}(0) = +\frac{1}{2}$, $S_0^{(B)}(0) = -\frac{1}{2}$ (only S_0 is 'damaged'). Since $S_x^{(A)}(t) \geqslant S_x^{(B)}(t)$ for all x and $t > 0$, the amount of damage at any times $t > 0$ is related to response and correlation functions. More precisely, the response function is given by the average value

$$R(x,t) = \langle |S_x^{(A)}(t) - S_x^{(B)}(t)| \rangle \tag{2}$$

while the static correlation function is given by

$$C(x) = \lim_{t\to\infty} \langle |S_x^{(A)}(t) - S_x^{(B)}(t)| \rangle \tag{3}$$

where now $S_0^{(A)}(t) = +\frac{1}{2}$, $S_0^{(B)}(t) = -\frac{1}{2}$ are kept fixed for all times (they are not kept fixed in (2), and in all cases considered in the following). This implies, in particular, that any damage has to heal at all temperatures above T_c and at any non-zero magnetic field B. Moreover, at $T < T_c$ any finite initial damage cannot modify the spontaneous symmetry breaking and must also heal due to ergodicity and mixing within each broken state.

For other updating algorithms, response and correlation functions are given by (2) and (3) *without* the absolute values. Without monotonicity the average damage (which is still defined with absolute values) is then larger than the response and correlation functions, respectively [11]. In particular, this means that the damage spreading transition for $B = 0$ must be at a temperature $T_d \leqslant T_c$. In three dimensions this was indeed seen for Glauber updating by Costa [12] and Le Caër [13, 14] who found $T_d/T \approx 0.96$ and ≈ 0.91, respectively. The latter author also found that for each $T > T_d$ there is a finite critical magnetic field $B_d(T)$ above which the damage no longer spreads. For $d = 2$ there seems to be no published study of the influence of magnetic fields, while the behaviour at $B = 0$ was studied in [3]. These authors concluded that $T_d = T_c$.

One of the reasons for the present study was to check whether there really is a qualitative difference between two and three dimensions. None of the above estimates of T_d were very precise, and it could well be that $T_d = T_c$ for all dimension, or $T_d < T_c$ for all d. The latter case would be particularly interesting. Due to the conjecture of [6], we should then expect that the behaviour near T_d is governed by directed percolation. Notice that this should be true in any case for $B_d \neq 0$, provided such transitions do actually show up at $T > T_d$. To test this universality is the other main purpose of this paper.

In all our studies we used the multispin code described in [6] which allowed us to study simultaneously 32 or 64 lattices. We always started with partially ordered lattices where all spins were uncorrelated. We also used completely independent initial configurations for the different replicas. Thus from each run we could estimate the damage in $32 \times \frac{31}{2}$ or $64 \times \frac{63}{2}$ pairs of replicas, respectively. Actually, in this way we measured not the *spreading* but the *healing* of damage, but this should be equivalent. In particular, the necessary scaling laws for DP are well known both for spreading and for healing. For $B = 0$, starting with $M_0 \neq 0$ is then essential just to break symmetry. In addition, a bad choice of M_0 can lead to very late scaling. Our data shown were obtained after some experimenting with the precise value of M_0, but no systematic optimization was done. Lattice sizes were up to 2579×2580 and $309^2 \times 310$ with helical boundary conditions, and updating was done in parallel on checker-board sublattices.

In all runs, the only measured quantity was the total amount of damage $D(t)$ (= the Hamming distance between the replicas) as a function of time. If indeed the transition is in the DP universality class, we should expect power-law decays at T_d,

$$D(t) \sim t^{-\delta} \tag{4}$$

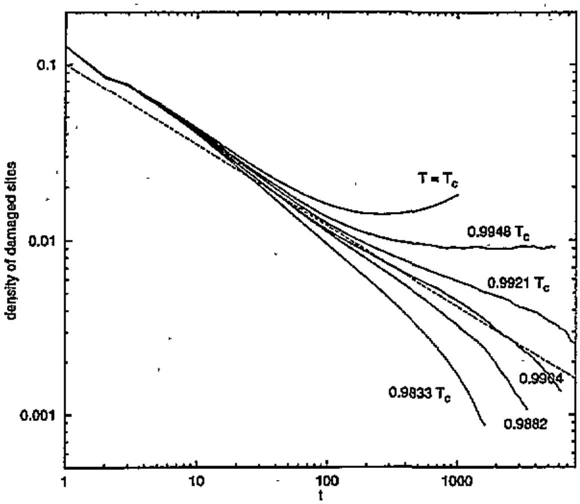

Figure 1. Log–log plot of the total damage $D(t)$ (divided by the lattice volume) in the 2D Ising model with Glauber dynamics. Temperatures are as indicated in the figure. In the initial states all spins were independent, and the average magnetization was $M_0 = \frac{3}{4}$. The broken line (whose intercept is arbitrary) has the slope -0.46 predicted from DP.

with $\delta = 0.46$ in $d = 2$ [15] and $\delta = 0.74$ in $d = 3$ [16]. Away from T_d, we should see a scaling law

$$D(t) \sim t^{-\delta}\psi\big((T - T_d)t^{1/\nu_t}\big) \tag{5}$$

with $\psi(0)$ finite and with $\nu_t = 1.29$ ($d = 2$) and 1.12 ($d = 3$), respectively.

Results for $D(t)$ as a function of time are shown in figure 1 for seven different values of T. We see clearly that the transition occurs below T_c. Roughly, we find $T_d/T_c = 0.992 \pm 0.002$. A more precise estimate is difficult due to the obvious deviations from scaling; indeed, these deviations are not surprising. They result from the closeness between T_c and T_d. At our estimated value of T_d we have $\epsilon \equiv (T_c - T_d)/T_c \approx 8 \times 10^{-3}$, and thus an Ising correlation time $\tau \approx \epsilon^{-\nu_t} \approx 3 \times 10^4$. Thus we have no chance to see DP scaling in our present simulations. One might hope to improve the situation by starting with equilibrium states, but the change would only be marginal. The Ising correlation length would be $\xi \approx \epsilon^{-\nu} \approx 125$. Using $\nu/\nu_t = 0.57$ for DP in $(2+1)$ dimensions, we can estimate that DP scaling should be seen only for $t \gg \xi^{\nu_t/\nu} \approx 5000$.

Things improve if we go to $T > T_c$. In figure 2 we show analogous results for $T = 5 = 2.203T_c$, for four different values of B. This time we should be far enough away from the Ising transition so that the DP scaling should set in much earlier. This is indeed seen very clearly, and our exponent δ is obviously in agreement with the prediction of DP. We also made runs at values of B further away from B_d (not shown in figure 2), and they confirmed that ν_t is compatible (within rather large errors) with the DP value.

We thus felt justified to also use the decay of the damage to locate B_d for other values of T. More precisely, we estimated B_d as that field strength at which (4) holds for $500 < t < 2000$ with the exponent δ predicted by DP. Results of this analysis are shown in figure 3. We see, in particular, that B_d tends towards a finite value for $T \to \infty$. In this limit the amplitude in (4) diverges (since the decay of $D(t)$ sets in only at times $>$ constant $\times\ T$), but we find DP exponents for all finite T.

Finally, we also made the same analyses for $d = 3$. The results for $B = 0$ (shown in

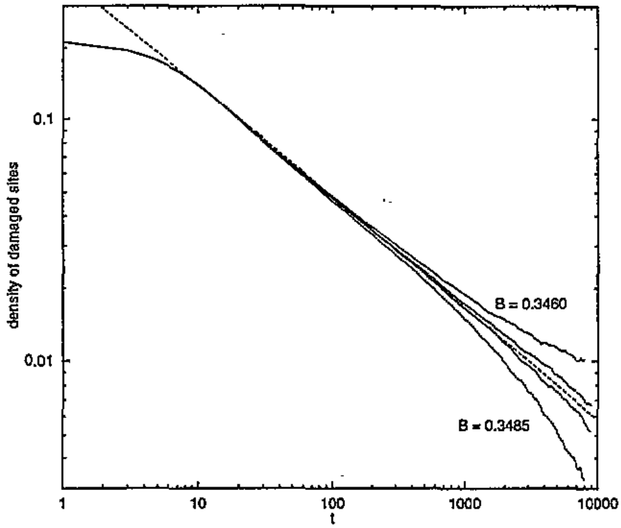

Figure 2. Similar to figure 1, but for fixed $T = 5$ and for four different values of the magnetic field: $B = 0.346, 0.347, 0.3475$ and 0.3485. Initial magnetization was $M_0 = \frac{1}{8}$.

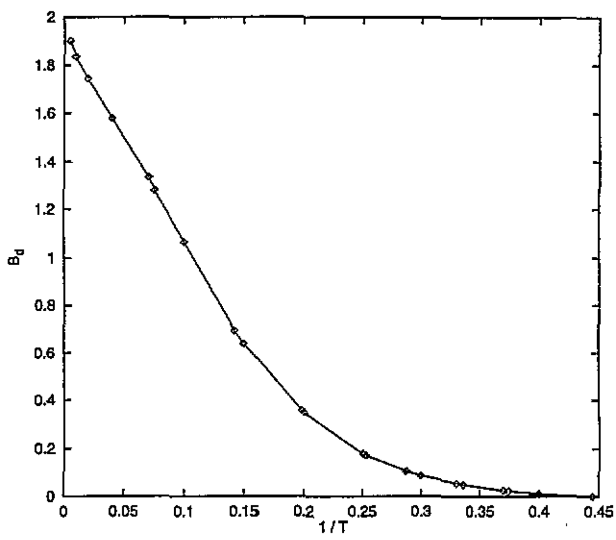

Figure 3. Critical magnetic field against $\beta = 1/T$, for $d = 2$. Damage can only spread below the curve, while it always heals for $B > B_{\mathrm{d}}$.

figure 4) are in better agreement with DP than those for $d = 2$, as we should have expected from the larger distance of T_{d} from the Curie point,

$$T_{\mathrm{d}}/T_{\mathrm{c}} = 0.9225 \pm 0.0005 \tag{6}$$

(we use $T_{\mathrm{c}} = 0.221\,655$ [17]). From figure 4 we see that both δ and ν_t agree with the values from DP. On the other hand, our precise value of T_{d} indicates that it is not identical to the percolation transition for minority spins in Ising correlated percolation [18] where $T_{\mathrm{p}} = 0.96$ with an error < 0.01. It is also slightly different from the damage spreading transition in the $\pm J$ spin glass [19] and from the percolation threshold for saturated bonds

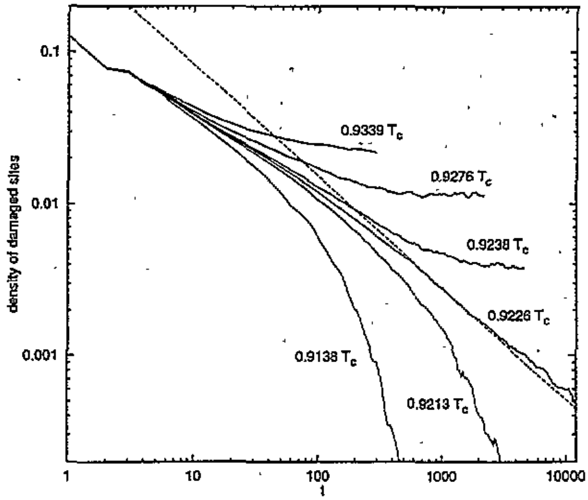

Figure 4. Similar to figure 1, but for $d = 3$. The broken line has slope -0.74.

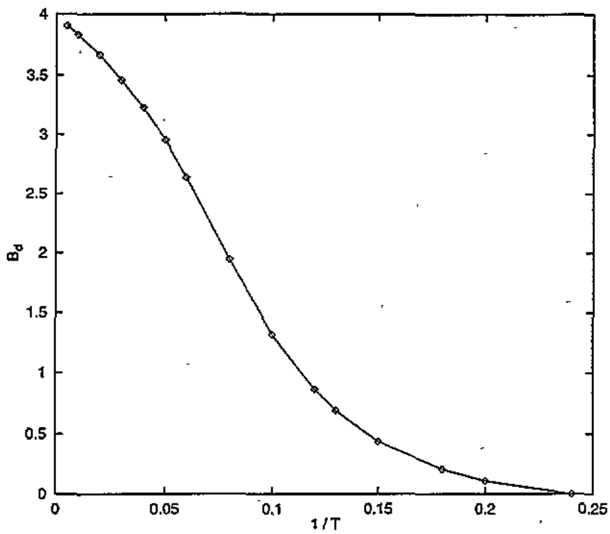

Figure 5. Similar to figure 3, but for $d = 3$.

in the $\pm J$ spin glass [20].

Results for B_d in three dimensions are shown in figure 5. They are in good agreement with the less precise values of [14]. For all temperatures, the observed exponent δ agreed at B_d with that of DP.

Since there exists a DP transition in $1 + 1$ dimensions, we might also expect to find a damaging transition in the $1 - d$ Ising model. Numerically we found no such transition. For $B = 0$ damage spreads at all temperatures, while it heals at all T for $B \neq 0$. We do not have any theoretical explanation for this. It does not, of course, contradict our conclusion that the transition is DP in higher dimensions.

In summary we have shown that damage spreading in the Ising model with Glauber dynamics is qualitatively the same in two and three dimensions, contrary to previous findings. Our values for T_d are much more precise than previous estimates, as are also our values of B_d for $T > T_d$. Part of this is due to an efficient implementation of multispin coding which uses the fact that we *wanted* to simulate several lattices with identical sequences of random numbers (otherwise providing independent random numbers is one of the problems in this sort of multispin coding [21]). But the main feature of the present investigation was that we directly measure the amount of damage as a function of time in runs with large initial damage. This also allowed us to compare our results most directly with predictions of directed percolation. We found that they are all compatible with the damage spreading transition being in the DP universality class, though the deviations from the predicted scaling laws in some cases are very large. But in those cases the origin of the deviations is well understood, and the deviations should disappear in the (very much delayed) scaling limit.

Our results thus support the conjecture that damage spreading transitions are generically (i.e. when they do not coincide with transitions of the undamaged system) in the DP universality class. This stresses that one should again study damage spreading in disordered systems (spin glasses [20, 19, 22], Kauffman models [23, 1, 24]) and check whether they are in the class of DP with frozen randomness [25]. Notice that there is a slight problem for Kauffman models where it can happen that damage neither spreads nor heals. But it should be possible to modify these models such that all non-spreading damages do heal. That this might then lead to the DP behaviour with frozen randomness was suggested earlier in [26].

I am indebted to I Campbell, N Jan, M Schreckenberg and D Stauffer for correspondence and discussions which I found extremely helpful. This work was supported by Deutsche Forschunggemeinschaft, SFB 237.

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